g) Formulate stability inequalities by using Sylvester's criteria to insure the positive-definiteness of the matrix δ given by

$$\delta = \begin{bmatrix} \alpha & \beta \\ \beta^T & \gamma \end{bmatrix}$$

where

$$\alpha = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_3 \end{bmatrix} \qquad \beta = \begin{bmatrix} \beta_{21}\beta_{22}\cdots\beta_{2n} \\ \beta_{31}\beta_{32}\cdots\beta_{3n} \end{bmatrix}$$

$$\gamma = \begin{bmatrix} \gamma_{11}\gamma_{12}\cdots\gamma_{in} \\ \gamma_{21} \\ \vdots \\ \gamma_{n1} \end{bmatrix}$$

h) Formulate "isolation conditions" by determining the circumstances under which the simple spin under consideration is an isolated quasi-gyrostatic motion in the following sense: let 0 be the origin of the $2(n+1)$ dimensional vector space described in Sec. VII of Ref. 1. Then a simple spin is an isolated quasi-gyroscopic motion if there exists a neighborhood of 0 in which 0 is the only point corresponding to a quasi-gyroscopic motion.

The last paragraph of Ref. 1 now describes the conclusions that can be drawn regarding the stability of a simple spin, provided d and e of Ref. 1 be replaced with g, while f of Ref. 1 is replaced with h.

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Derivatives of Eigenvalues and Eigenvectors for a General Matrix

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I. Introduction

IN the optimal design of systems where the dynamic stability and/or response of the system is a function of several parameters it is often desirable to know the derivatives of the eigenvalues and eigenvectors of a characteristic equation of the system. Rudisill and Bhatia[1,2] utilized the first and second derivatives of the eigenvalues of the equations for steady-state oscillation of an aircraft structure to find the first and second derivatives of the flutter velocity. These derivatives of the flutter velocity were used in gradient search procedures to find the minimum mass structure for a given flutter velocity.

For self-adjoint systems Wittrick[3] derived expressions for the first derivatives of the eigenvalues, and Fox and Kapoor[4] derived expressions for the first derivatives of the eigenvectors. For nonself adjoint-systems Rogers[5] derived expressions for the first derivative of the eigenvalues and the eigenvectors. Rudisill and Bhatia,[1,2] and Plaut and Huseyin[6] derived expressions for the first derivatives of the eigenvalues and eigenvectors and the second derivatives of the eigenvalues. Garg[7] developed a method for finding the first derivatives of the eigenvalues and eigenvectors which requires the solution of $2(n+1)$ equations for an n degree of freedom system.

In the references previously cited the expressions for any one of the derivatives of the eigenvectors of a nonself-adjoint system requires (with the exception of Garg's[7] method) the value of all of the left-hand and right-hand eigenvectors. For large degree-of-freedom systems the expense of finding all of these eigenvectors may be excessive when the derivatives of only one of the eigenvectors are needed. It is the purpose of this Note to derive expressions for the derivatives of the eigenvalues and eigenvectors which are expressions of only one left-hand and one right-hand eigenvector. The method may be extended to find any order of derivative of the eigenvalue and eigenvector.

II. Analysis

Consider the matrix equation

$$(A-\lambda_i B)U_i = 0 \tag{1}$$

where A and B are $n\times n$ matrices and $\lambda_i(i=1,2,\ldots,n)$ are assumed to be distinct eigenvalues of the characteristic equation

$$\det|A-\lambda_i B| = 0 \tag{2}$$

and U_i is the right-hand eigenvector corresponding to λ_i. The left-hand eigenvector V_i corresponding to λ_i is such that

$$V_i'(A-\lambda_i B) = 0 \tag{3}$$

where the prime denotes the transpose of the matrix.

Taking the partial derivative of Eq. (1) with respect to the jth parameter yields the relation

$$(A_{,j}-\lambda_{i,j}B-\lambda_i B_{,j})U_i+(A-\lambda_i B)U_{i,j} = 0 \tag{4}$$

The first partial derivative of λ_i may now be found by premultiplying Eq. (4) by the transpose of the left-hand eigenvector V_i, then substituting Eq. (3) into the results and finally solving for the partial derivative

$$\lambda_{i,j} = V_i'(A_{,j}-\lambda_i B_{,j})U_i/(V_i'BU_i) \tag{5}$$

The second partial derivative of the eigenvalue λ_i may be found by differentiating Eq. (4) with respect to the kth parameter, premultiplying by the transpose of the left-hand eigenvector V_i, making use of Eq. (3) and then solving for the second partial derivative

$$\begin{aligned}\lambda_{i,jk} = [&V_i'(A_{,jk}-\lambda_{i,j}B_{,k}-\lambda_{i,k}B_{,j}-\lambda_i B_{,jk})U_i+ \\ &V_i'(A_{,j}-\lambda_{i,j}B-\lambda_i B_{,j})U_{i,k}+ \\ &V_i'(A_{,k}-\lambda_{i,k}B-\lambda_i B_{,k})U_{i,j}]/(V_i'BU_i)\end{aligned} \tag{6}$$

Rudisill and Bhatia,[1,2] and Paut and Huseyin[8] developed expressions similar to Eqs. (5) and (6) for the first and second derivatives, they also developed expressions for the derivatives of the eigenvectors which were functions of all n of the eigenvectors U_i and V_i. Next a method will be developed for finding the derivative of U_i without prior knowledge of the other $n-1$ right- and left-hand eigenvectors.

If all of the eigenvalues of Eq. (2) are distinct then there are n distinct linearly independent eigenvectors U_i. The rank of $A-\lambda_i B$ is then $n-1$ and there are only $n-1$ components of U_i which are unique and from Eq. (4) there are only $n-1$ components of $U_{i,j}$ which are unique. Since the length of an eigenvector is arbitrary, the components of the eigenvector may be rendered unique by imposing the constraint

$$U_i'U_i = 1 \tag{7}$$

then

$$U_i'U_{i,j} = 0 \tag{8}$$

Equations (4) and (8) may be combined to form the relations

$$\left[\frac{A-\lambda_i B}{U_i'}\right]U_{i,j} = -\left[\frac{A_{,j}-\lambda_{i,j}B-\lambda_i B_{,j}}{0}\right]U_i \tag{9}$$

Received October 23, 1973. This research was supported by NASA Research Grant NGR-41-001-027. The author wishes to express his appreciation to R. V. Doggett Jr. of the NASA Langley Research Center for his advice on the preparation of this Note.

Index categories: Structural Dynamics Analysis; Structural Design, Optimal.

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where the coefficients of $U_{i,j}$ and U_i are $n+1$ by n partitioned matrices. After premultiplying Eq. (9) by the transpose of $[(A-\lambda_i B)/U_i']$ the following matrix equation may be formed

$$CU_{i,j} = -EU_i \tag{10}$$

where

$$C = [A' - \lambda_i B' \vdots U_i] \left[\begin{array}{c} A - \lambda_i B \\ \hline U_i' \end{array}\right] \tag{11}$$

$$E = [A' - \lambda_i B' \vdots U_i] \left[\begin{array}{c} A_{,j} - \lambda_{i,j} B - \lambda_i B_{,j} \\ \hline 0 \end{array}\right] \tag{12}$$

where C and E are $n \times n$ matrices and $-EU_i$ is a column vector.

The derivatives of the eigenvector U_i may be found by solving the n Eqs. (10) for the components of $U_{i,j}$ after U_i and $\lambda_{i,j}$ have been substituted into Eq. (10).

Solutions to Eq. (10) may be conveniently found by decomposing matrix C into two or more matrices by means of a Choleski decomposition or some other decomposition method. The decomposed matrices would not contain derivatives; thus, if m first partial derivatives of U_i are desired with respect to m variable parameters, it is necessary to perform the decomposition only one time and then use the decomposed matrices in a forward and backward substitution scheme to solve for the components of each of the vectors $U_{i,j}$ ($j = 1, 2, \ldots, m$).

After evaluating $\lambda_{i,j}$ and $U_{i,j}$ from Eqs. (5) and (10), respectively, their values may be substituted into Eq. (6) where the value of $\lambda_{i,jk}$ may be determined. It should be noted that only one right-hand eigenvector U_i and one left-hand eigenvector V_i is required to determine $\lambda_{i,jk}$; however, this advantage is offset somewhat by the requirement that a set of n simultaneous equations must be solved.

The second derivative of the eigenvectors may be found by differentiating Eq. (4) with respect to the kth variable parameter and rearranging the resulting expression in the form

$$\begin{aligned}(A - \lambda_i B)U_{i,jk} = {} & -(A_{,jk} - \lambda_{i,jk} B - \lambda_{i,j} B_{,k} - \\ & \lambda_{i,k} B_{,j} - \lambda_i B_{,jk}) U_i - (A_{,j} - \lambda_{i,j} B - \lambda_i B_{,j}) U_{i,k} - \\ & (A_{,k} - \lambda_{i,k} B - \lambda_i B_{,k}) U_{i,j}\end{aligned} \tag{13}$$

or

$$(A - \lambda_i B) U_{i,jk} = -G \tag{14}$$

Differentiating Eq. (8) with respect to the kth parameter yields the relation

$$U_i' U_{i,jk} = -U_{i,k}' U_{i,j} \tag{15}$$

Equations (14) and (15) represent an overdetermined system of $n+1$ equations and n unknown components of $U_{i,jk}$. These equations may be written in the form

$$\left[\begin{array}{c} A - \lambda_i \\ \hline U_i' \end{array}\right] U_{i,jk} = -\left[\begin{array}{c} G \\ \hline U_{i,k}' U_{i,j} \end{array}\right] \tag{16}$$

Premultiplying Eq. (16) by the transpose of the coefficient matrix of $U_{i,jk}$ yields n simultaneous equation in n unknowns, i.e.,

$$CU_{i,jk} = -\left[\begin{array}{c} A - \lambda_i \\ \hline U_i' \end{array}\right]' \left[\begin{array}{c} G \\ \hline U_{i,k}' U_{i,j} \end{array}\right] \tag{17}$$

where the matrix C is the same as in Eq. (11). Equation (17) may be solved for $U_{i,jk}$ by using the same procedure by which Eq. (10) is solved for $U_{i,j}$ provided that all of the derivatives on the right-hand side of Eq. (13) are known.

The procedure described previously may be continued to find any order of derivative of λ_i and U_i provided the derivatives exist. The expressions for finding the derivatives of eigenvalues and eigenvectors for nonself-adjoint systems may be applied to self-adjoint systems by setting $V_i = U_i$.

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An Experiment on an Impulse Loaded Elastic Ring

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THE response of a circular elastic ring to a cosine distributed impulse over half the ring circumference has been the subject of analytical[1,2] and experimental[3,4] investigations. Closed form solutions for the membrane and bending stresses are presented in Ref. 2, and the experimental work reports measurements of membrane strains produced by magnetically propelled flyer plates[3] and direct magnetic pressure pulses.[4] For the flyer plate technique, energy from a capacitor bank is used to propel a thin metallic plate away from a rigid backup mass and onto the structural ring. The gap between flyer and ring is sized such that the capacitor bank is rung down before impact. However, ring deflection calculations presented in the Appendix of this Note demonstrate that the gap between ring and backup mass must also be sized to permit ring bending displacements. That is, the gap must be large enough to permit the ring to flex without hitting the backup mass. The experimental arrangement for the magnetic pressure loading technique[4] uses a rigid backup mass and does not permit free flexural motion.

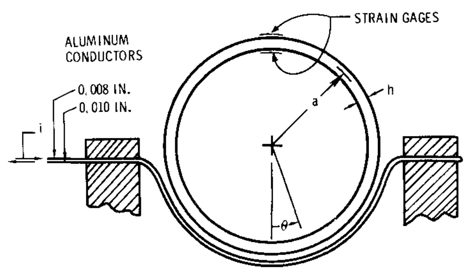

Fig. 1 Experimental arrangement.

Received October 29, 1973; revision received December 6, 1973. This work was supported by the U.S. Atomic Energy Commission. The authors thank M. J. Sagartz for helpful suggestions and E. E. Jones for performing the experiments.

Index category: Structural Dynamic Analysis.

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